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Certain Bibasic Hypergeometric Transformation Formulae and Their Application to Rogers–Ramanujan Identities

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In this paper, a variety of generalized bibasic hypergeometric transformation formulae have been investigated with the objective of finding a multiple series generalized Rogers–Ramanujan type of identity. It has been shown that some very interesting new multiple series identities of the Rogers–Ramanujan type can be found where there is more than one infinite product in which the terms in each product advance in different powers of q.

1. INTRODUCTION

During the last two decades, a variety of generalized hypergeometric identities have been investigated in an attempt to find a generalized Rogers–Ramanujan type of identity. In almost all such generalizations, there has been recourse to some auxiliary function. Andrews [2], Bressoud [4, 5], and others have also introduced their auxiliary functions containing two bases, which are often connected by a power of q. Recently, we [8] have obtained a very general bibasic hypergeometric transformation formula with two independent bases and have shown that a number of known results are special cases of this transformation.

In this paper, we obtain another bibasic hypergeometric transformation formula with the help of the transformation [8, Eq. (15)] and shall discuss a number of special cases involving a multiple series Rogers–Ramanujan type of identity.

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We shall use the following notations throughout this paper. For $|q| < 1$, let

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ (1-a)\cdots(1-aq^{n-1}), & n \geq 1, \end{cases}$$

$$((a_r);q)_n = (a_1,\dots,a_r;q)_n = (a_1;q)_n \cdots (a_r;q)_n,$$

$$(a;q)_\infty = \prod[a;q] = \prod_{n=1}^{\infty}(1-aq^{n-1}), \prod\begin{bmatrix} a;q \\ b \end{bmatrix} = \prod[a;q] \Big/ \prod[b;q].$$

An ${}_r\phi_s$ basic hypergeometric series is defined as

$${}_r\phi_s\begin{bmatrix} a_1,\dots,a_r;q;z \\ b_1,\dots,b_s \end{bmatrix} = \sum_{n=0}^{\infty} \frac{(a_1,\dots,a_r;q)_n z^n}{(q,b_1,\dots,b_s;q)_n} \left\{(-1)^{nq\binom{n}{2}}\right\}^{1+s-r}$$

with $\binom{n}{2} = n(n-1)/2$.

Also, for $M_i = m_i + m_{i+1} + \cdots + m_{k-1}$, m, r, and k are positive integers, let

$$U_{(m_{k-1})}(a,b,c,(a_{2k-3});q)$$
$$= \frac{(aq/bc;q)_{m_{k-1}}(aq/a_1a_2;q)_{m_{k-2}} \cdots (aq/a_{2k-5}a_{2k-4};q)_{m_1}}{(aq/b,aq/c;q)_{M_{k-1}}(aq/a_1,aq/a_2;q)_{M_{k-2}} \cdots}$$
$$\times \frac{(a_1,a_2;q)_{M_{k-1}} \cdots (a_{2k-5},a_{2k-4};q)_{M_2}}{\cdots (aq/a_{2k-5},aq/a_{2k-4};q)_{M_1}(a_1a_2)^{M_{k-1}} \cdots (a_{2k-5}a_{2k-4})^{M_2}},$$

$$V_n(\lambda,(b_{2m});q_1) = \frac{(b_1,\dots,b_{2m};q_1)_n q_1^{(m-1)n}}{(\lambda q_1/b_1,\dots,\lambda q_1/b_{2m};q_1)_n (b_1 \cdots b_{2m})^n}, \qquad |q_1| < 1,$$

$$K_n(a,a_{2k-3};q) = \frac{(a_{2k-3};q)_n (a;q)_{2n}}{(aq/a_{2k-3};q)_n (a_{2k-3})^n}, \qquad J_n(a;\lambda) = \frac{\lambda^{(m+k/r)n}}{a^{(k-1)n}}.$$

Further, let

$$A_n(a;q) = \frac{(q^{-p};q)_n(-a)^n q^{n(n+1)/2}}{(q;q)_n}, \qquad p \geq 0,$$

and

$$B_n(c,(a_{2k-3});q) = \frac{(c,a_1,\dots,a_{2k-3};q)_n(ca_1 \cdots a_{2k-3})^{-n}}{(aq/c,aq/a_1,\dots,aq/a_{2k-3};q)_n}.$$

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We shall first prove the following general bibasic transformation formula from which identities of the Rogers–Ramanujan type can be deduced:

THEOREM 1.

$$(aq;q)_\infty \sum_{n,m_1,\dots,m_{k-1}\geq 0} \frac{U_{(m_{k-1})}(a,b,c,(a_{2k-3});q^2)V_{m_1}(\lambda,(b_{2m});q_1)}{(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-1}}}$$

$$\times \frac{K_{M_1}(a,a_{2k-3};q^2)q^{M_1^2+3M_1+2(M_2+\cdots+M_{k-1})+n^2-pn}}{(q;q)_{n-2M_1}(aq;q)_{n+2M_1}(-1)^{M^1}J_{M_1}^{-1}(a;\lambda)a^{-(n+M_2+\cdots+M_{k-1})}}$$

$$\times \sum_{s\geq 0} \frac{\left(aq^{4M_1},a_{2k-3}q^{2M_1};q^2\right)_s V_s\left(\lambda q_1^{2M};\left(b_{2M}q_1^{M_1}\right);q_1\right)}{\left(q^2,aq^{2+2M_1}/a_{2k-3};q^2\right)_s(1-\lambda)}$$

$$\times \frac{\left(1-\lambda q_1^{2M_1+2s}\right)\left(q^{2M_1-n};q\right)_{2s}q^{(3-2M_1+2n)s}}{\left(aq^{1+2M_1+n};q\right)_{2s}\left(a_{2k-3}\right)^s J_s^{-1}(a;\lambda)}$$

Proof. In the general bibasic transformation formula [8, Eq. (15)] replacing first q by q^2 and then setting $a_{2k-2}=q^{-n}$ and $a_{2k-1}=q^{1-n}$, we have

$$\sum_{s\geq 0} \frac{(a,b;q^2)_s B_s(c,(a_{2k-3});q^2)V_s(\lambda,(b_{2m});q_1)\left(1-\lambda q_1^{2s}\right)}{(q^2,aq^2/b;q^2)_s b^s(1-\lambda)}$$

$$\times \frac{\lambda^{(m+k/r)s}q^{2s^2+2ks}}{(aq;q)_{n+2s}(q;q)_{n-2s}}$$

$$= \sum_{j=0}^{p} A_j(a;q) \sum_{n\geq 0} \frac{(a,b;q^2)_n V_n(\lambda,(b_{2M});q_1)}{(q^2,aq^2/b;q^2)_n}$$

$$\times \frac{B_n(c,(a_{2k-3});q^2)\left(1-\lambda q_1^{2n}\right)q^{6n^2+2kn-2pn+4nj}}{(1-\lambda)b^n a^{-(k+1)n}J_n^{-1}(a;\lambda)}. \tag{1}$$

$$= \sum_{M_1,\dots,M_{k-1}\geq 0} \frac{U_{(M_{k-1})}(a,b,c,(a_{2k-3});q^2)V_{M_1}(\lambda,(b_{2M});q_1)}{(q^2;q^2)_{M_1}\cdots(q^2;q^2)_{M_{k-1}}}$$

$$\times \frac{K_{M_1}(a,(a_{2k-3});q^2)\,q^{M_1^2+3M_1+2(M_2+\cdots+M_{k-1})}}{(q;q)_{n-2M_1}(aq;q)_{n+2M_1}(-1)^{M_1}J_{M_1}^{-1}(a;\lambda)a^{-(M_2+\cdots+M_{k-1})}}$$

$$\times \sum_{s \geq 0} \frac{\left(aq^{4M_1}a_{2k-3}q^{2M_1}; q^2\right)_s V_s\left(\lambda q_1^{2M_1}; \left(b_{2M}q_1^{M_1}\right); q_1\right)}{\left(q^2, aq^{2+2M_1}/a_{2k-3}; q^2\right)_s (1-\lambda)}$$

$$\times \frac{\left(1-\lambda q_1^{2M_1+2s}\right)\left(q^{2M_1-n}; q\right)_{2s} q^{(3-2M_1+2n)s}}{\left(aq^{1+2M_1+n}; q\right)_{2s} \left(a_{2k-3}\right)^s J_s^{-1}(a;\lambda)}. \tag{2}$$

Now, in Bailey's transform [3], we set

$$u_s = (q;q)_s^{-1}, \qquad v_s = (aq;q)_s^{-1},$$

$$\alpha_{2s+1} = 0, \qquad \alpha_{2s} = \frac{(a,b;q^2)_s B_s\left(c,(a_{2k-3});q^2\right) V_s\left(\lambda,(b_{2m});q_1\right)}{\left(q^2, aq^2/b; q^2\right)_s b^s (1-\lambda)}$$

$$\times\left(1-\lambda q_1^{2s}\right)\lambda^{(m+k/r)s} q^{2s^2+2ks},$$

and

$$\delta_s = (x,y;q)_s \left(aq^{1-p}/xy\right)^s,$$

and evaluate $\langle \beta_n \rangle$ and $\langle \gamma_n \rangle$ by transformation (2) and the following transformation due to Verma [11]:

$${}_2\phi_1\left[\begin{matrix} x, y; q; \dfrac{\alpha c}{xy} \\ \alpha \end{matrix}\right] = \prod \left[\begin{matrix} \dfrac{\alpha}{x}, \dfrac{\alpha}{y}; q \\ \alpha, \dfrac{\alpha}{xy} \end{matrix}\right] {}_3\phi_2\left[\begin{matrix} x, y, c; q; q \\ \dfrac{xyq}{\alpha}, 0 \end{matrix}\right] \tag{3}$$

(where x, y, or c is of the form q^{-p}. In the case $c = q^{-p}$, then (3) is valid only if $|\alpha c/xy| < 1$). We easily get (1) after some simplification.

3. PARTICULAR CASES

Let us take $\lambda = a$, $r = 1$, replace q_1 by q^2, and then make

$$b_1, \ldots, b_m \to \infty,$$
$$b_{m+1}, \ldots, b_{2m} \to 0$$

in (1). After summing the innermost series by a well-poised ${}_6\phi_5$ summation formula [6, Eq. (2.7.1)], we get

$$\frac{(aq;q)_\infty (aq/a_{2k-3};q)_\infty}{(aq^2/a_{2k-3};q^2)_\infty}$$
$$\times \sum_{n,m_1,\dots,m_{k-1}\ge 0} \frac{U_{(m_{k-1})}(a,b,c,(a_{2k-3});q^2)}{(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-1}}(aq;q^2)_n}$$
$$\times \frac{a^{M_1+\cdots+M_{k-1}+n} q^{M_1^2+M_1+2(M_2+\cdots+M_{k-1})+n^2-pn}(a_{2k-3})^{-M_1}}{(q;q)_{n-2M_1}(aq/a_{2k-3};q)_n (aq^{1+2n-2M_1}/a_{2k-3};q^2)_\infty}$$
$$= \sum_{j=0}^{p} A_j(a;q) \sum_{n\ge 0} \frac{(a,b;q^2)_n B_n(c,(a_{2k-3});q^2)(1-aq^{4n})}{(q^2,aq^2/b;q^2)_n(1-a)b^n}$$
$$\times a^{(k+2)n} q^{6n^2+2kn-2pn+4nj-2n}. \tag{4}$$

If we now make

$$b,c,a_1,\dots,a_{2k-3}\to\infty$$

in (4), we obtain the transformation

$$(aq;q)_\infty \sum_{n,m_1,\dots,m_{k-1}\ge 0} \frac{a^{n+M_1+\cdots+M_{k-1}} q^{n^2-pn+2(M_1^2+\cdots+M_{k-1}^2)}}{(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-1}}(aq;q^2)_n(q;q)_{n-2M_1}}$$
$$= \sum_{j=0}^{p} A_j(a;q) \sum_{n\ge 0} \frac{(a;q^2)_n(1-aq^{4n})}{(q^2;q^2)_n(1-a)} (-1)^n a^{(k+2)n}$$
$$\times q^{(2k+5)n^2-2pn+4nj-n}. \tag{5}$$

Again, if we first take $b=-q\sqrt{a}$ and then make

$$c,a_1,\dots,a_{2k-3}\to\infty$$

in (4), we get the transformation

$$(aq;q)_\infty \sum_{n,m_1,\dots,m_{k-1}\ge 0} \frac{a^{M_1+\cdots+M_{k-1}+n} q^{2(M_1^2+\cdots+M_{k-1}^2)}}{(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-1}}(aq;q^2)_n}$$
$$\times \frac{q^{n^2-pn}}{(-q\sqrt{a};q^2)_{M_{k-1}}(a;q)_{n-2M_1}}$$
$$= \sum_{j=0}^{p} A_j(a;q) \sum_{n\ge 0} \frac{(a;q^2)_n(1-aq^{4n})}{(q^2;q^2)_n(1-a)} (-1)^n a^{(k+3/2)n}$$
$$\times q^{(2k+4)n^2-2pn+4nj-n}. \tag{6}$$

Identities of the Rogers–Ramanujan Type

Let us take $p = 0$ with $a = 1$ and q^2 in (5) and then make use of Jacobi's triple product identity [6, Eq. (1.6.1)]:

$$\sum_{n=-\infty}^{\infty} (-1)^n z^n q^{n^2} = \prod_{n=1}^{\infty} \left(1 - zq^{2n-1}\right)\left(1 - q^{2n-1}/z\right)\left(1 - q^{2n}\right). \quad (7)$$

We get two known identities [12, Eqs. (4.12) and (4.13)] due to Verma and Jain. If we now take $p = 1$ with $a = 1$ and q^2 in (5) and use the above identity (7), we thus obtain the following two identities, respectively, which are believed to be new:

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2) + (4n-2)M_1 - n}}{(q;q)_n (q^2;q^2)_{m_1} \cdots (q^2;q^2)_{m_{k-3}} (q;q^2)_{n+2M_1}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-2) (\mathrm{mod}\, 4k+2)}}^{\infty} (1 - q^n)^{-1} + \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2k (\mathrm{mod}\, 4k+2)}}^{\infty} (1 - q^n)^{-1}, \quad (8)$$

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2) + (4n+4)M_1 + n + 2(M_2 + \cdots + M_{k-3})}}{(q;q)_n (q^2;q^2)_{m_1} \cdots (q^2;q^2)_{m_{k-3}} (q;q^2)_{n+2M_1+1}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 4 (\mathrm{mod}\, 4k+2)}}^{\infty} (1 - q^n)^{-1}. \quad (9)$$

It may be noted that, for $k = 3$ and 4, (8) reduces to the identities [10, Eqs. (60) and (61)] and [9, Eq. (25)] while (9) reduces to the identities [10, Eq. (60)] and [9, Eq. (27)] due to Slater and Singh, respectively.

Further, let us take $p = 2$ with $a = q^2$ in (5) and then use identity (7). We have the following identity, which is also believed to be new:

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 4nM_1 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2 + M_1 + \cdots + M_{k-3})}}{(q;q)_n (q^2;q^2)_{m_1} \cdots (q^2;q^2)_{m_{k-3}} (q;q^2)_{n+2M_1+1}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 6 (\mathrm{mod}\, 4k+2)}}^{\infty} (1 - q^n)^{-1} + q \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 (\mathrm{mod}\, 4k+2)}}^{\infty} (1 - q^n)^{-1}. \quad (10)$$

For $k = 3$ and 4, it reduces to the known identities [10, Eqs. (59) and (61)] and [9, Eq. (28)] due to Slater and Singh, respectively.

Let us now take $a = 1$ with $p = 0$, 1, and 2 in (6) and make use of identity (7). We get, respectively, the following three identities:

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 4nM_1 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2)} (q; q^2)_{n+2M_1}^{-1}}{(q; q)_n (q^2; q^2)_{m_1} \cdots (q^2; q^2)_{m_{k-3}} (-q; q^2)_{M_{k-3}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-1) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1}, \tag{11}$$

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2) + (4n-2)M_1 - n} (q; q^2)_{n+2M_1}^{-1}}{(q; q)_n (q^2; q^2)_{m_1} \cdots (q^2; q^2)_{m_{k-3}} (-q; q^2)_{M_{k-3}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-3) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1} + \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-1) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1}, \tag{12}$$

and

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2) + (4n-4)M_1 - 2n} (q; q^2)_{n+2M_1}^{-1}}{(q; q)_n (q^2; q^2)_{m_1} \cdots (q^2; q^2)_{m_{k-3}} (-q; q^2)_{M_{k-3}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-5) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1} + \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-3) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1} + q^{-1}(1+q) \prod_{\substack{n=1 \\ n \not\equiv 0, \pm(2k-1) (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1} \tag{13}$$

(cf. Bressoud [5, Eq. (3.6)]).

Last, if we take $a = q^2$ with $p = 0$, 1, and 2 in (6) and then make use of identity (7), we have the following identities, respectively:

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{n^2 + 6M_1^2 + 2(M_2^2 + \cdots + M_{k-3}^2) + 4nM_1} (q; q^2)_{n+2M_1+1}^{-1}}{(q; q)_n (q^2; q^2)_{m_1} \cdots (q^2; q^2)_{m_{k-3}} (-q^2; q^2)_{M_{k-3}}} = \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 2 (\mathrm{mod}\, 4k)}}^{\infty} (1 - q^n)^{-1}, \tag{14}$$

$$\sum_{n,m_1,\ldots,m_{k-3}\ge 0} \frac{q^{n^2+6M_1^2+(4n+4)M_1+2(M_2^2+\cdots+M_{k-3}^2+M_2+\cdots+M_{k-3})}}{(q;q)_n(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-3}}(-q^2;q^2)_{M_{k-3}} \times (q;q^2)_{n+2M_1+1}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 4 \pmod{4k}}}^{\infty} (1-q^n)^{-1}, \tag{15}$$

and

$$\sum_{n,m_1,\ldots,m_{k-3}\ge 0} \frac{q^{n^2+6M_1^2+2(M_2^2+\cdots+M_{k-3}^2+M_2+\cdots+M_{k-3})}}{(q;q)_n(q^2;q^2)_{m_1}\cdots(q^2;q^2)_{m_{k-3}}(-q^2;q^2)_{M_{k-3}}} \times \frac{q^{(un+2)M_1}}{(q;q^2)_{n+2M_1+1}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 6 \pmod{4k}}}^{\infty} (1-q^n)^{-1} + q \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 2 \pmod{4k}}}^{\infty} (1-q^n)^{-1} \tag{16}$$

(cf. Bressoud [5, Eqs. (3.8) and (3.7)]).

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We shall now prove the following general bibasic transformation formula from which identities of the Rogers–Ramanujan type can also be deduced:

THEOREM 2.

$$(a^2q^2;q^2)_\infty \sum_{n,m_1,\ldots,m_{k-1}\ge 0} \frac{U_{(m_{k-1})}(a,b,c,(a_{2k-3});q)V_{m_1}(\lambda,(b_{2m});q_1)}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}}$$

$$\times \frac{K_{M_1}(a,a_{2k-3};q)\,q^{(M_1/2)(M_1+3)+M_2+\cdots+M_{k-1}+2n^2-2pn}}{(q^2;q^2)_{n-M_1}(a^2q^2;q^2)_{n+M_1}(-1)^{M_1}J_{M_1}^{-1}(a;\lambda)\,a^{-(M_2+\cdots+M_{k-1}+2n)}}$$

$$\times \sum_{s\ge 0} \frac{(aq^{2M_1},a_{2k-3}q^{M_1};q)_s V_s(\lambda q_1^{2m_1},(b_{2m}q_1^{M_1});q_1)}{(q,aq^{1+M_1}/a_{2k-3};q)_s(1-\lambda)}$$

$$\times \frac{\left(1-\lambda q_1^{2M_1+2s}\right)\left(q^{2M_1-2n};q^2\right)_s q^{(2-M_1+2n)s}}{\left(a^2q^{2M_1+2n+2};q^2\right)_s\left(-a_{2k-3}\right)^s J_s^{-1}(a;\lambda)}$$

$$= \sum_{j=0}^{p} A_j\left(a^2;q^2\right)\sum_{n\geq 0}\frac{(a,b;q)_n V_n\left(\lambda,(b_{2m});q_1\right)}{(q,aq/b;q)_n}$$

$$\times \frac{B_n\left(c,(a_{2k-3});q^2\right)\left(1-\lambda q_1^{2n}\right)q^{3n^2+kn-2pn+4nj}}{(1-\lambda)b^n a^{-(k+1)n}J_n^{-1}(a;\lambda)}. \tag{17}$$

Proof. On taking $a_{2k-2} = -q^{-n}$ and $a_{2k-1} = q^{-n}$ in the general bibasic transformation formula [18, Eq. (15)], we obtain the following result:

$$\sum_{s\geq 0}\frac{(a,b;q)_s B_s\left(c,(a_{2k-3});q\right)\left(1-\lambda q_1^{2s}\right)V_s\left(\lambda,(b_{2m});q_1\right)}{(q,aq/b;q)_s(1-\lambda)b^s}$$

$$\times \frac{\lambda^{(m+k/r)s}q^{s^2+ks}}{\left(q^2;q^2\right)_{n-s}\left(a^2q^2;q^2\right)_{n+s}}$$

$$= \sum_{m_1,\ldots,m_{k-1}\geq 0}\frac{U_{(m_{k-1})}\left(a,b,c,(a_{2k-3});q\right)V_{M_1}\left(\lambda,(b_{2m});q_1\right)}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}}$$

$$\times \frac{K_{M_1}\left(a,a_{2k-3};q\right)q^{(M_1/2)(M_1+3)+M_2+\cdots+M_{k-1}}}{\left(q^2;q^2\right)_{n-m_1}\left(a^2q^2;q^2\right)_{n+m_1}(-1)^{M_1}J_{M_1}^{-1}(a;\lambda)a^{-(M_2+\cdots+M_{k-1})}}$$

$$\times \sum_{s\geq 0}\frac{\left(aq^{2M_1},a_{2k-3}q^{M_1};q\right)_s V_s\left(\lambda q_1^{2M_1},\left(b_{2M}q_1^{M_1}\right);q_1\right)}{\left(q,aq^{1+M_1}/a_{2k-3};q\right)_s(1-\lambda)}$$

$$\times \frac{\left(1-\lambda q_1^{2M_1+2s}\right)\left(q^{2M_1-2n};q^2\right)_s q^{(2-M_1+2n)s}}{\left(a^2q^{2+2M_1+2n};q^2\right)_s\left(-a_{2k-3}\right)^s J_s^{-1}(a;\lambda)}. \tag{18}$$

Now, in Bailey's transform [3], choose

$$u_s = \left(q^2;q^2\right)_s^{-1}, \qquad v_s = \left(a^2q^2;q^2\right)_s^{-1},$$

$$\alpha_0 = 1, \qquad \alpha_s = \frac{(a,b;q)_s B_s\left(c,(a_{2k-3});q\right)V_s\left(\lambda,(b_{2m});q_1\right)}{(q,aq/b;q)_s(1-\lambda)b^s}$$

$$\times\left(1-\lambda q_1^{2s}\right)\lambda^{(m+k/r)s}q^{s^2+ks},$$

and

$$\delta_s = \left(x,y;q^2\right)_s\left(a^2q^{2-2p}/xy\right)^s.$$

We evaluate $\langle \beta_n \rangle$ and $\langle \gamma_n \rangle$ by the transformations (18) and (3), respectively, and we get (17) after some simplification.

5. PARTICULAR CASES

Let us now take $\lambda = a$, $r = 1$, and $q_1 = q$ and then make

$$b_1, \dots, b_m \to \infty,$$

$$b_{m+1}, \dots, b_{2m} \to 0$$

in (17). On summing the innermost series on the left by a well-poised ${}_6\phi_5$ summation formula [6, Eq. (2.7.1)], we obtain the following transformation:

$$\begin{aligned}
&\frac{(aq;q)_\infty}{(aq/a_{2k-3};q)_\infty} \\
&\quad \sum_{n,m_1,\dots,m_{k-1}\geq 0} \frac{U_{(m_{k-1})}(a,b,c,(a_{2k-3});q)(a_{2k-3};q)_{M_1}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}} \\
&\quad \times \frac{(-aq^{1+2n};q)_\infty (a^2q^{2+2n}/a_{2k-3}^2;q^2)_\infty}{(q^2;q^2)_{n-M_1}(-aq^{1+2n-M_1}/a_{2k-3};q)_\infty} \\
&\quad \times \frac{q^{(M_1/2)(M_1+1)+M_2+\cdots+M_{k-1}+2n^2-2pn}}{a^{-(M_1+\cdots+M_{k-1}+2n)}(-a_{2k-3})^{M_1}} \\
&\qquad = \sum_{j=0}^{p} A_j(a^2;q^2) \sum_{n\geq 0} \frac{(a,b;q)_n B_n(c,(a_{2k-3});q)(1-aq^{2n})}{(q,aq/b;q)_n(1-a)b^n} \\
&\qquad\quad \times a^{(k+2)n} q^{3n^2+(k-2p+4j-1)n}. \qquad (19)
\end{aligned}$$

If we make

$$b, c, a_1, \dots, a_{2k-3} \to \infty$$

in (19), we easily get

$$\begin{aligned}
&(a^2q^2;q^2)_\infty \\
&\quad \times \sum_{n,m_1,\dots,m_{k-1}\geq 0} \frac{a^{M_1+\cdots+M_{k-1}+2n} q^{M_1^2+\cdots+M_{k-1}^2}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}} \times \frac{q^{2n^2-2pn}}{(q^2;q^2)_{n-M_1}(-aq;q} \\
&\qquad = \sum_{j=0}^{p} A_j(a^2;q^2) \sum_{n\geq 0} \frac{(a;q)_n(1-aq^{2n})}{(q;q)_n(1-a)} (-1)^n a^{(k+2)n} \\
&\qquad\quad \times q^{(2k+5)(n^2/2)-(n/2)+(4j-2p)n}. \qquad (20)
\end{aligned}$$

Further, if we take $b = -\sqrt{aq}$ and then make

$$c, a_1, \ldots, a_{2k-3} \to \infty$$

in (19), we obtain

$$\begin{aligned}
&(aq;q)_\infty \sum_{n,m_1,\ldots,m_{k-1}\geq 0} \frac{(-aq^{1+2n};q)_\infty a^{M_1+\cdots+M_{k-1}+2n}}{(q;q)_{m_1}\cdots(q;q)_{m_{k-1}}\left(-\sqrt{aq};q\right)_{M_{k-1}}} \\
&\quad\times \frac{q^{M_1^2+\cdots+M_{k-1}^2+2n^2-2pn}}{(q^2;q^2)_{n-M_1}} \\
&\quad= \sum_{j=0}^{p} A_j(a^2;q^2) \sum_{n\geq 0} \frac{(a;q)_n(1-aq^{2n})}{(q;q)_n(1-a)}(-1)^n a^{(k+3/2)n} \\
&\quad\quad\times q^{(k+2)n^2+(-2p+4j-1/2)n}. \qquad (21)
\end{aligned}$$

Identities of the Rogers–Ramanujan Type

Let us take $p = 0$ with $a = 1$ and q in (20) and make use of identity (7). We get two known identities [12, Eqs. (4.7) and (4.8)] due to Verma and Jain. If we now take $p = 1$ with $a = 1$ and q in (20) and use identity (7), we obtain the following identities which seem to be new:

$$\begin{aligned}
&(-q;q)_\infty \sum_{n,m_1,\ldots,m_{k-3}\geq 0} \frac{q^{M_2^2+\cdots+M_{k-3}^2+3M_1^2+2n^2-2n+(4n-2)M_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{2n+2M_1}} \\
&= \prod_{\substack{n=1\\ n\not\equiv 0,\pm(k-2)(\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1} + \prod_{\substack{n=1\\ n\not\equiv 0,\pm(k-1)(\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1}, \qquad (22)
\end{aligned}$$

$$\begin{aligned}
&(-q;q)_\infty \sum_{n,m_1,\ldots,m_{k-3}\geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2+2n^2+M_1+\cdots+M_{k-3}+4nM_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{2n+2M_1+1}} \\
&= \prod_{\substack{n=1\\ n\not\equiv 0,\pm 3(\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1} - q \prod_{\substack{n=1\\ n\not\equiv 0,\pm 1(\mathrm{mod}\ 2k+1)}}^{\infty} (1-q^n)^{-1}. \qquad (23)
\end{aligned}$$

Further, if we take $p = 2$ with $a = 1$ and q in (20) and then make use of identity (7), we have the following two interesting identities, respectively:

$$(-q;q)_\infty \sum_{n,m_1,\dots,m_{k-3}\geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2+2n^2-4n+(4n-4)M_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{2n+2M_1}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm(k-3)(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1}$$

$$+ \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm(k-4)(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1} + q^{-2}(1+q^2)$$

$$\times \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm k(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1}, \tag{24}$$

$$(-q;q)_\infty$$

$$\times \sum_{n,m_1,\dots,m_{k-3}\geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2+2n^2-M_1+M_2+\cdots+M_{k-3}-2n+4nM_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{2n+2M_1+1}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 5(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1} + (1+q^2)$$

$$\times \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 1(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1} - q \prod_{\substack{n=1\\ n\not\equiv 0,\,\pm 3(\mathrm{mod}\,2k+1)}}^{\infty} (1-q^n)^{-1}. \tag{25}$$

Lastly, on taking $a = q$ with $p = 0$, 1, and 2 in (21) and then using identity (7), we obtain the following identities, respectively:

$$\sum_{n,m_1,\dots,m_{k-3}\geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2+2n^2+3M_1+M_2+\cdots+M_{k-3}+2n+4nM_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{M_{k-3}}}$$

$$\frac{(-q^{2+2n+2M_1};q)_\infty}{(-q;q)_{M_{k-3}}}$$

$$= \prod_{\substack{n=1\\ n\not\equiv 0\ \pm 1(\mathrm{mod}\,2k)}}^{\infty} (1-q^n)^{-1}, \tag{26}$$

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2+2n^2+M_1+\cdots+M_{k-3}+4nM_1}(-q^{2+2n+2M_1};q)_\infty}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}(-q;q)_{M_{k-3}}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3 (\mathrm{mod}\, 2k)}}^{\infty} (1-q^n)^{-1} - q \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 1 (\mathrm{mod}\, 2k)}}^{\infty} (1-q^n)^{-1}, \tag{27}$$

$$\sum_{n, m_1, \ldots, m_{k-3} \geq 0} \frac{q^{3M_1^2+M_2^2+\cdots+M_{k-3}^2-M_1+M_2+\cdots+M_{k-3}+2n^2-2n+4nM_1}}{(q^2;q^2)_n(q;q)_{m_1}\cdots(q;q)_{m_{k-3}}}$$

$$\frac{(-q^{2+2n+2M_1};q)_\infty}{(-q;q)_{M_{k-3}}}$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 5 (\mathrm{mod}\, 2k)}}^{\infty} (1-q^n)^{-1} + (1+q^2) \prod_{\substack{n=1 \\ n \not\equiv ,0, \pm 1 (\mathrm{mod}\, 2k)}}^{\infty} (1-q^n)^{-1}$$

$$- q \prod_{\substack{n=1 \\ n \not\equiv 0, \pm 3 (\mathrm{mod}\, 2k)}}^{\infty} (1-q^n)^{-1}. \tag{28}$$

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